Notes on Jacobian method

September 24, 2012

Let's begin with a very simple 3x3 game:

	A	B	C
A	1	0	0
B	0	1	0
C	0	0	1

We will express population proportions in terms of p, q, and r which represent the probability of A, B, and C respectively. Populations will be represented with the vector $< p, q, r >$.

The replicator dynamics for this game is given by:

$$\dot{p} = p(u(A, < p, q, r >) - u(< p, q, r >, < p, q, r >))$$
$$\dot{q} = q(u(B, < p, q, r >) - u(< p, q, r >, < p, q, r >))$$
$$\dot{r} = r(u(C, < p, q, r >) - u(< p, q, r >, < p, q, r >))$$

Substituting in the actual payoffs from the game with becomes:

$$\dot{p} = p(p - p^2 - q^2 - r^2)$$
$$\dot{q} = q(q - p^2 - q^2 - r^2)$$
$$\dot{r} = r(r - p^2 - q^2 - r^2)$$

There are seven Nash equilibria of this game. They are $< 1, 0, 0 >$, $< 0, 1, 0 >$, $< 0, 0, 1 >$, $< 1/2, 1/2, 0 >$, $< 0, 1/2, 1/2 >$, $< 1/2, 0, 1/2 >$, and $< 1/3, 1/3, 1/3 >$. We will just focus on two, one of the monomorphic population states (everyone plays A) and the totally mixed population state $< 1/3, 1/3, 1/3 >$.

Step 1: Computing the Jacobian

The Jacobian matrix is defined by taking each of the three equations, and taking the partial derivative with respect to each of the variables. This creates nine equations (one for each pair of equation and variable) which are arranged in a matrix.

$$J = \begin{pmatrix} \frac{\partial}{\partial p}\dot{p} & \frac{\partial}{\partial q}\dot{p} & \frac{\partial}{\partial r}\dot{p} \\ \frac{\partial}{\partial p}\dot{q} & \frac{\partial}{\partial q}\dot{q} & \frac{\partial}{\partial r}\dot{q} \\ \frac{\partial}{\partial p}\dot{r} & \frac{\partial}{\partial q}\dot{r} & \frac{\partial}{\partial r}\dot{r} \end{pmatrix}$$

For this particular game this becomes:

$$\begin{pmatrix} -p^2 + (1-2p)p + p - q^2 - r^2 & -2pq & -2pr \\ -2pq & -p^2 - q^2 - r^2 + q + q(1-2q) & -2qr \\ -2pr & -2qr & -p^2 - q^2 - r^2 + r + r(1-2r) \end{pmatrix}$$

Step 2a: Calculate the eigenvalues the Jacobian for $< 1, 0, 0 >$

First let's consider the Jacobian at the point $< 1, 0, 0 >$. At this point, the Jacobian becomes,

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So by the Hartman-Grobman theorem, we know that the system near this equilibrium point behaves like the linear system of differential equations:

$$\dot{x} = -x$$

$$\dot{y} = -y$$

$$\dot{z} = -z$$

It should already be clear that the point $< 0, 0, 0 >$ in this system is asymptotically stable, but just to be sure we can calculate the eigenvalues of the Jacobian. They are $\{-1, -1, -1\}$. Since they are all negative, we know that this point must be stable.

Step 2b: Calculate the eigenvalues the Jacobian for $< 1/3, 1/3, 1/3 >$

Now lets' consider the Jacobian at the point $< 1/3, 1/3, 1/3 >$. At this point, the Jacobian is,

$$\begin{pmatrix} \frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \end{pmatrix}$$

We will skip the step of writing down the linear system of differential equations, and just go straight to the eigenvalues. They are, $\{-1/3, 1/3, 1/3\}$. Because some are positive, we can conclude the system is unstable.